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over-estimation of the second stimulus. Let r' be the stimulus that under conditions of over-estimation is judged equal to r; r_+' the stimulus that under the same conditions is just noticeably greater than r' and r_0' the analogue r_0. We shall define a particular case of zero difference of intensity by

Post. 10.— $I_r - I_r' = 0.$

We may now solve this equation for r . The solution is particularly simple when $r_0' = r_0$, and as these may be anticipated to be very close-lying quantities, we might for practical purposes identify them with their mean $\bar{r}_0$.
Writing for brevity

$$\frac{r_+ - \bar{r}_0}{r - \bar{r}_0} = k$$

we readily obtain

$$I_r = \left(\log \frac{k}{k'}\right)^{-1} \cdot \log \frac{r - \bar{r}_0}{r' - \bar{r}_0}.$$

The empirical question of the truth of Weber's law depends upon our ability to find conditions under which the subjective parameters k, r_0, r_ω remain constant. This is only a special case of the more general experimental problem, How do these parameters vary with the conditions of the experiment in which r is made to vary?

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THE ISSUES CONCERNING MATERIAL IMPLICATION

IN a recent article in this JOURNAL,[1] Mr. Norbert Wiener objects to criticisms which I have made of the relation of material implication as the basis of inference. Mr. Wiener's paper fails to join the significant issue, but this may be as much my fault as his. If others have understood my papers as he seems to have done, then in justice to myself, as well as to those who are interested in Mr. Russell's logic, I should attempt some restatement. And the spirit of Mr. Wiener's criticism is so eminently just and fair that I take pleasure in pointing out my agreement with many of his contentions. For the sake of brevity and clearness, the matters to be considered may be summarized in set theses.

1. The relation of material implication, $p \supset q$, which figures in the mathematical logic in *Principia Mathematica*, is not the relation which we ordinarily have in mind when we say that q can be inferred from p.

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[1] Vol. XIII., p. 656.

2. There is no sound reason why we should abandon our usual meaning of "implies" for this unusual one. The difference between material implication and ordinary "inference" does not point to any inadvertance or shortcoming of our logical sense; it points rather to the novelty—as an "implication"—of the relation $p \supset q$.

3. The unusual character of this relation is responsible for the presence, in any system of symbolic logic which uses it, of certain "peculiar" theorems, *e. g.*, $\sim p \supset (p \supset q)$, "A false proposition implies any proposition," and $\sim(p \supset q) \supset p$, "If p does not imply q, then p is true."[2]

4. Theorems (and postulates) in a calculus of propositions are used in two ways within the system itself; (*a*) as premises *from* which further theorems are derived; (*b*) as rules, or principles of inference, *by* which theorems are derived. In any system of material implication, the theorems which we have spoken of as "peculiar" admit of valid use as premises. But they can not profitably be used as rules of inference, either within or outside of the system.

This thesis needs elucidation. The theorems in question *can be* used as rules of inference, and there is nothing in the nature of the system of material implication, as developed in *Principia Mathematica*, to prevent their being so used.[3] But their use in this way gives such propositions as $p \supset (q \supset q)$, "If p is true, then q implies q," and $\sim[(pq) \supset q] \supset q$, "If '$p$ and q are both true' does not imply q, then q is true." When used as rules of inference outside the system, these "peculiar" theorems give such propositions as, "To-day is Monday implies $2 + 2 = 4$," and "Socrates was a solar myth, implies that to-day is Monday." It is characteristic of the propositions so obtained that they assert a "logical" relation between statements which are likely to be irrelevant. Very few of the "peculiar" theorems are even printed in *Principia Mathematica*,[4] although any number of them can be derived from the postulates. Their presence in the system does not, in practise, affect the proofs in later sections of that work.

5. The system of "strict implication" which has been presented

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[2] These theorems are the result of using the "principle of addition," $p \supset (p > q)$, and the definition $(p \supset q) = (\sim p \vee q)$, in the same proof. Whenever this procedure produces a theorem which can not be otherwise proved, this theorem belongs to the class I speak of as "peculiar."

[3] The method by which a theorem, or postulate, is used as a principle of inference in *Principia Mathematica* is as follows: The proposition in question must have as its main, or asserted, relation, an implication, $\supset$. Other symbolic propositions are substituted for *p*, or *q*, *etc.*, wherever *p*, or *q*, *etc.*, appear in this proposition, and in such wise that what precedes the asserted $\supset$ becomes identical with some previous proposition. What *follows* this $\supset$, is then separately asserted as a new lemma or theorem.

[4] See *2·02, *2·21, *2·24, *2·5, *2·51, *2·52, *2·521, and *3·4.

in my papers is such that none of the "peculiar" theorems of material implication can be proved from its postulates, but all the *other* theorems of material implication can be so proved (supposing, of course, each material implication replaced by the corresponding strict implication).[5]

6. Therefore, a distinguishing mark of material implication as a species in the genus "possible systems of symbolic logic" is the inclusion of these "peculiar" theorems.

7. By neglecting ever to use these theorems as rules of inference, one may find the system of material implication a sufficiently workable organon of proof, because in making this convenient omission, one uses precisely those principles which are *not* peculiar to material implication, but conform to the ordinary meaning of "implies."

The above theses are sufficient to establish the fact that the system of material implication is defective, in this sense: it contains any number of theorems—theorems about "implication"—which can never be of the slightest value as principles of inference. And it is precisely by the inclusion of these useless theorems that it differs from another, equally consistent, system of symbolic logic. These are the more important considerations; and if I read Mr. Wiener correctly, he does not deny anything so far stated, though he might desire to offer extenuation for what I have called the "defect" of material implication.

We now approach more difficult matters, and care should be taken not to "read into" the theses which follow anything which is not stated.

8. The system of material implication, as developed in *Principia Mathematica,* is derived from precisely formulated postulates and definitions, by operations which are clearly stated and rigorously adhered to. In exactly that sense, it is a mathematically consistent system.

9. The assumptions are true of *their own* meaning of "implies," and the theorems are true in the same sense.

10. The assumptions and theorems are such that whoever grants the truth of the assumptions can not consistently deny the truth of the theorems. The theorems are relevant, in content, to the postulates: the theorems ("peculiar" ones not excepted) *can be,* each and every one, inferred from the postulates.[6]

Mr. Wiener seems to think that I deny some one or more of the last three theses,—I am not sure which ones.

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[5] This statement is not quite accurate, because, in developing the system of "strict implication," I made an error which Mr. Wiener does not note. But that error is easily corrected.

[6] See "The Matrix Algebra for Implications," this JOURNAL, Vol. XI., esp. p. 598.

11. It *is not demonstrated* in *Principia Mathematica* that the theorems of the system of material implication can be inferred from the postulates.

Mr. Wiener devotes the larger part of his paper to the discussion of this thesis. Unfortunately, he seems to confuse "p can be inferred from q" with "p *is* inferred from q," and with "It is *demonstrated that* p can be inferred from p."[7] If every proposition which can be inferred from another were, by that fact alone, demonstrated to be so inferrable, then "proof" would be unnecessary. Also, he seems inclined to fall back upon the assertion that the theorems are derived from the postulates by (mathematically or symbolically) rigorous operations.[8] This would be to beg the question whether a derivation based upon material implication is, or can be, inference.

The *pros* and *cons* of this matter can be presented most succinctly in the form of a dialogue between *X* and myself. *X*'s sentiments may not be Mr. Wiener's; and any rhetorical advantage which I may take of my straw-man is not intended for him. But I try to state both sides fairly: the main object is elucidation of the issue. At best, this is not easy, and the reader should have constantly in mind the character of material implication. "p materially implies q" means exactly: "At least one of the two propositions, 'p is false' and 'q is true,' is a true proposition;" or "It is false that p is true and q false." If p is false, then p materially implies q; if q is true, then p materially implies q. Of any two true propositions—however irrelevant in content—each materially implies the other. Of any two false propositions, each materially implies the other. Of any two propositions one of which is true and the other false, the false one materially implies the true one. Of any two propositions chosen at random, it is certain in advance that at least one of them materially implies the other.

X. It is demonstrated in *Principia Mathematica* that if the postulates of the system of material implication are true, then the theorems are true.

L. Really, you beg the question with your "if . . ., then . . .," which may mean a connection of the kind we call "logically necessary" between the content of the "if" clause and the "then" clause, or may indicate only a relation of material implication, such as, "If to-day is Monday, then $2 + 2 = 4$."

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[7] See his paper, the last sentence beginning on page 658. This is Mr. Wiener's final word on the topics with which we are concerned. The remainder of his article is given over to discussion of certain related matters and incidental criticism of a view which is not mine. He attributes it to me on the basis of one sentence, the context of which he ignores, and a footnote which he has misinterpreted.

[8] On page 657, where one of the two issues is stated to be: "Is the coherence and self-consistence which it [the Russellian logic] claims for itself, and tries to justify by an orderly derivation of its theorems from a small and simple set of postulates, genuine or factitious?"

X. It means a material implication, but that makes no difference to the argument. The nature of material implication is such that if it holds between postulates and theorems and the postulates are true, then the theorems are true. And *that* is the criterion of valid inference.

L. I might ask you whether this "if . . ., then . . ." is a material implication, but I refrain. As a fact, it is a strict implication.[9]

X. Then you grant my last statement?

L. I grant that the postulates of material implication are true, of their own meaning of "implies," and since true propositions do not materially imply false ones, the theorems are true in the same sense. Still I must point out that true postulates may materially imply theorems which are *quite irrelevant.* That is what the "Too Brief Set of Postulates"[10] was written to show. Does that paper prove that a true postulate may materially imply theorems which are true but irrelevant?

X. In a way, it does. The assumption in that case is a "postulate" by courtesy only; but I grant that you can write down a true assumption which will materially imply irrelevant propositions. However, that is not the point: the important thing is that material implication can never hold between true postulates and *false* theorems.

L. I have said that true propositions do not materially imply false ones.

X. Then you haven't a leg left to stand on.

L. Wait a bit. Are the postulates of material implication demonstrated in *Principia Mathematica?*

X. Of course not; but they are explained so that anybody can see they are true.

L. They are true, as I have said, but are they demonstrated?

X. No.

L. Then in what sense is the truth of the theorems which they materially imply demonstrated?

X. In the sense that, whether the postulates are true or not, if you *grant* the postulates, you must grant the theorems.

L. I am sorry if I seem to split hairs, but your statement is not sufficiently precise. We "grant" assumptions in two senses,—either because we believe they are true and are ready to believe what they imply, or for the sake of the argument and to discover their logical consequences. "Postulates" are usually granted in the second sense, are they not? Do you use "grant" in this sense?

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[9] The postulates of the "Matrix Algebra" give the theorem, $[(p \supset q)p] \cup q$, where $\supset$ indicates a material implication, but $\cup$ is a strict implication.

[10] This Journal, Vol. XII., p. 523.

X. It is not of the smallest consequence.

L. It would not be if we were concerned with the ordinary meaning of "implies," but material implication is peculiar in that respect. Let me ask you: Do you mean that, whether the postulates of material implication are, in fact, true or not, if they *were* true, the theorems *would necessarily* be true?

X. There is no mysterious physical or causal connection in material implication—or any other kind—if that is what you are driving at.

L. But tell me: do you admit that "Socrates was a solar myth" materially implies $2+2=5$?

X. Yes; but only because Socrates was *not* a solar myth.

L. Quite so. But if Socrates *were* a solar myth, would it be true that $2+2=5$? If you granted some paradoxer his assumption that Socrates was a solar myth, would you feel constrained to go on and grant that $2+2=5$?

X. I suppose you mean to harp on "irrelevant" some more.

L. I mean precisely this: The postulates of the system of material implication in *Principia Mathematica* are not demonstrated, and the "proofs" of the theorems show only that they are materially implied by the postulates. Suppose, for the moment, the postulates were not true. Then not only might the theorems be false, but—for anything that is *demonstrated*—they might be such that if the postulates *were* true they would *still* be false.

X. What of it, since the postulates are, in fact, true?

L. Only this: the relation which is established between postulates and theorems, is the relation which *is established,*—whether the postulates are true or not does not alter *that.* And the relation which is established between the postulates and theorems of the system in question is precisely the same relation which you admit to hold between "Socrates was a solar myth" and $2+2=5$. Perhaps I am a purist, but I should never call the establishing of a connection like that "demonstration" or "proof."

X. But you know well enough that the postulates and theorems of the system of material implication are relevant to each other.

L. Yes, I know that they are, in fact, relevant. As I have said, the theorems can be inferred from the postulates. Once more, the question is not whether they *can be* inferred, but whether this is *demonstrated?* You see the question always comes back to the proper meaning of "implies" which ought to figure in demonstration and proof. "Proof" requires that a connection of content or meaning or logical significations be established. And this is not done for the postulates and theorems in material implication: what is more, it can not be done without calling on principles outside the system.

X. Let me see if I understand you. You grant that the relation of material implication is such, and the procedure of the early sections of *Principia Mathematica* is such, that the postulates being, in fact, true, and the fact that they materially imply the theorems being established, it follows that the theorems are true?

L. Yes.

X. And when you say that the theorems are not demonstrated, all you mean is that they are not shown to be relevant in content to the postulates?

L. Precisely that. You think that this is a very little "all"; I think it is a big "all." For a relation which does not indicate relevance of content is merely a connection of "truth values," not what we mean by a "logical" relation or "inference."

It is regrettable if my use of "demonstration" and "proof" has confused any one, but I think I have used the terms in their commonly accepted and proper signification.

And now I think the matter is concluded, fairly to all concerned.

X. But Mr. Russell has a new and better meaning of "demonstration." You do not even mention that. In the later sections—

L. May I not make my offering to the gods without reciting the whole *Principia Mathematica?*

And should we speak of new and better meanings, or of new and better analyses? "Implication" and "proof" are pretty fundamental categories. The contributions of Mr. Russell and Mr. Whitehead are important—none more so—and yet they will not thank us if we speak as though logic and mathematical demonstration were things which they invented and can do with as they please. We are always in need of clearing away confusions and realizing more clearly our own intentions. But where the basic things of human reason are concerned, genuinely novel meanings are justly regarded with suspicion.

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REVIEWS AND ABSTRACTS OF LITERATURE

The Human Worth of Rigorous Thinking. C. J. KEYSER. New York: Columbia University Press. 1916. Pp. 314.

This book is a collection of essays and addresses which have appeared in various periodicals during the last fifteen years. They concern a variety of topics, ranging from issues of such a metaphysical character as that of "The Existence of the Hypercosmic," and